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Noise spectrum of the quantum Langevin equation

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Abstract. It is shown that the spectral density of the Ford–Kac–Mazur noise for the quantum Langevin equation can be derived from the general requirement of the validity of the fluctuation–dissipation theorem.

In the quantum theory of an open system it is necessary to take into account simultaneously the quantum nature of the system and the dissipative interaction with the environment. This is a difficult and interesting problem which has been approached from a variety of points of view. The vast literature on the subject, which has been recently reviewed in a number of articles (Haken 1975, Hasse 1978, Messer 1979, Dekker 1981), reflects the diversity of the methods employed and the labour involved in establishing equivalencies or discrepancies among the different formulations.

The approach most close in spirit to the classical theory of Brownian motion is based on the quantum Langevin equation first derived by Ford, Kac and Mazur (FKM) from a microscopic model for the interaction of the Brownian particle with a thermal bath (Ford *et al* 1965). Considering a quantum particle of mass m subjected to the mechanical potential $V(q)$ in contact with a thermal bath at the temperature T, in the FKM theory the position and momentum operators $q(t)$, $p(t)$ in the Heisenberg representation obey the equations of motion

$$\dot{q} = p/m \tag{1a}$$

$$\dot{p} = -\partial V(q)/\partial q - fp + E(t) \tag{1b}$$

where f is a friction coefficient and $E(t)$ is a Gaussian operator stochastic process with the commutator

$$[E(t_1), E(t_2)] = \mathrm{i} f\hbar(\partial/\partial t_1 - \partial/\partial t_2)\delta(t_1 - t_2) \tag{2}$$

and the expectations

$$\langle E(t)\rangle = 0 \tag{3a}$$

$$\langle E(t_1)E(s)\rangle_s = \tfrac{1}{2}\langle E(t)E(s) + E(s)E(t)\rangle = \frac{mf}{\pi}\int_0^\infty \mathrm{d}\omega\, \hbar\omega \coth\left(\frac{\hbar\omega}{2kT}\right)\cos[\omega(t-s)]. \tag{3b}$$

In the FKM theory equations (1) are derived exactly from the microscopic model under the assumption of a special form of the interaction among the degrees of freedom of the thermal bath. Therefore the problem of the general validity of equations (1) remains open (Benguria and Kac 1981). In particular it may seem that the properties of the noise term $E(t)$ may be left to modelling (Alicki and Messer 1982, and references quoted therein).

In this paper we carry out explicitly the computation for the harmonic oscillator and we show that given the form (1) of the quantum Langevin equation, the properties of the noise are fixed by the general requirement of the validity of the fluctuation–dissipation theorem (FDT). Let us assume that the symmetrised covariance of the noise is given by

$$\langle E(t)E(s)\rangle_s = \Gamma(|t-s|) \tag{4}$$

where for the moment Γ is an unknown function. Assume also that the Brownian particle is an harmonic oscillator subjected to a time dependent external potential $V(q)=\frac{1}{2}K^2q^2-\lambda(t)q$ where K is the frequency of the oscillator and $\lambda(t)=\theta(-t)\lambda$ with θ the unit step function. Namely for $t<0$ the oscillator is in thermal equilibrium and is acted upon by the constant force λ. At $t=0$ the force is switched off and the relaxation toward the new equilibrium state corresponding to $\lambda=0$ takes place for $t>0$. Solving (1) for the average value of the position, for positive time we find for the relaxation function $\Phi(t)$

$$\Phi(t)=\langle q(t)\rangle/\lambda=(\mathrm{e}^{-ft/2}/K^2)[\cos \nu t+(f/2\nu)\sin \nu t] \tag{5}$$

where $\nu\equiv[K^2-(f/2)^2]^{1/2}$. To obtain (5) we have used (3a) and the initial conditions $\langle q(0)\rangle=\lambda/K^2$, $\langle p(0)\rangle=0$ which are the equilibrium values of position and momentum at $t=0$.

On the other hand, computing the equilibrium symmetrised correlation function with $\lambda=0$, from (1) one obtains

$$\begin{aligned} G(t-t') &= \langle q(t)q(t')\rangle_s \\ &= \int_0^t \mathrm{d}s \int_0^{t'} \mathrm{d}u\, \mathrm{e}^{-f(t-s)/2} \sin \nu(t-s)\, \mathrm{e}^{-f(t'-u)/2} \sin \nu(t'-u)\Gamma(|s-u|). \end{aligned} \tag{6}$$

Next turning to the Fourier transforms

$$\Phi(\omega)=(f/\pi)\{[(\tfrac{1}{2}f)^2+(\omega+\nu)^2][(\tfrac{1}{2}f)^2+(\omega-\nu)^2]\}^{-1} \tag{7}$$

$$G(\omega)=\Gamma(\omega)\{[(\tfrac{1}{2}f)^2+(\omega+\nu)^2][(\tfrac{1}{2}f)^2+(\omega-\nu)^2]\}^{-1} \tag{8}$$

and imposing the FDT requirement (Kubo 1957)

$$G(\omega)=\tfrac{1}{2}\hbar\omega \coth(\hbar\omega/2kT)\Phi(\omega) \tag{9}$$

we immediately find

$$\Gamma(\omega)=(f\hbar\omega/2\pi)\coth(\hbar\omega/2kT) \tag{10}$$

which is the spectral density of the symmetrised covariance appearing in (3b).

We conclude by noticing that the models for the quantum Brownian oscillator with a δ-correlated noise (Alicki and Messer 1982, Ruggiero and Zannetti 1982) of the type

$$\langle E(t)E(s)\rangle_s=\hbar Kf\coth(\hbar K/2kT)\delta(t-s) \tag{11}$$

are not at variance with the FDT but are expected to arise in the limit of weak coupling with the reservoir $\hbar f \ll kT<\hbar K$ (Ford *et al* 1965, Ullersma 1966).

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